PMM U.S.S.R., Vol.53, No.2, pp. 212-216, 1989
Printed in Great Britain

0021-8928/89 $10.00+0.00

ON THE THERMOCAPILLARY MOTION OF A DROP WITH HOMOGENEOUS INTERNAL HEAT EVOLUTION*

A.E. REDNIKOV and YU.S. RYAZANTSEV

An approximate analytical solution of the problem of the velocity and temperature distribution during the uniform motion of a drop with a homogeneous distribution of internal heat sources (sinks) has been obtained in the Stokes approximation at a small Péclet number. In the case of a linear dependence of the surface tension of the drop on temperature, it is shown that the thermocapillary effect associated with the motion of the drop can lead to the appearance of a tractive force as well as to a resistive force.

The existence of a temperature gradient on the surface of a drop gives rise to a gradient in the surface tension and may cause a motion of the drop. Examples of such a motion are known in the case when the asymmetry in the temperature distribution leading to the motion of the drop is determined by external sources and is not associated with the motion (see /1-4/, for example). A quite different situation, when an inhomogeneous temperature distribution on the surface of the drop is a consequence of its inherent motion, was first noted in /5/ in which a drop with a surface exo(endo) thermic reaction was considered. This problem was subsequently studied in /6/ and /7/.

In this paper we investigate a further example of the thermocapillary drift of a drop in which the asymmetry in the temperature field only arises on account of the motion of the drop and, in turn, brings about this motion.

Let a drop of a viscous incompressible liquid, within which distributed heat sources (sinks) of constant power uniformly act, be located in a viscous incompressible fluid which occupies the whole of space. The presence of heat sources (sinks) within a drop may be associated, for example, with the occurrence of a bulk chemical reaction or with the radioactive decay of the substance of which the drop is composed. It is obvious that, if the drop is at rest, the temperature at all points on its surface will be constant and the thermocapillary forces will be equal to zero. In order to establish the possibility of the abovementioned thermocapillary motion, let us consider the problem of the steady-state motion of a drop in the Stokes approximation. The solution of this problem enables one to find the velocity and temperature distribution in the drop and the medium surrounding it and also to obtain an expression for the force acting on the drop due to viscous and thermocapillary stresses. The difference in the magnitude of this force from the Stokes force determines the degree of development in the asymmetry of the temperature field and the termocapillary forces and the conditions for the existence of its thermocapillary drift.

In a reference frame associated with the centre of mass of the drop, the problem reduces to the stationary circumfluence of a drop by a planar parallel flow of a fluid. It is assumed that the drop maintains its spherical shape. The velocity and temperature distributions possess axial symmetry with respect to an axis passing through the centre of the drop in the direction of the velocity of the approach stream and a spherical coordinate system is therefore used in the analysis in which the radius r is measured from the centre of the drop while the angle θ is measured from the direction of the velocity of the approach stream. The densities, viscosities, thermal conductivities, and heat capacities of the substances comprising the drop and the external medium were assumed to be constant while the surface tension was assumed to be a linear function of the temperature.

Within the framework of the assumptions which have been formulated, the equations and the boundary conditions for the velocity and temperature outside of and within the drop can be written in the form

$$-\rho_i^{-1}\nabla p_i + \mathbf{g} + \nu_i\Delta \mathbf{v}_i = 0, \quad \operatorname{div}\mathbf{v}_i = 0$$
$$(\mathbf{v}_1\nabla)T_1 = \chi_1\Delta T_1$$
$$(\mathbf{v}_2\nabla)T_2 = \chi_2\Delta T_2 + Q, \quad Q = q/(c_{p2}\rho_2)$$

*Prikl. Matem. Mekhan., 53, 2, 271-277, 1989

$$r=a,\quad v_{r1}=v_{r2}=0,\quad v_{\theta 1}=v_{\theta 2}$$

$$\frac{p_2-p_1}{2}=\mu_2\frac{\partial v_{r2}}{\partial r}-\mu_1\frac{\partial v_{r1}}{\partial r}+\frac{\sigma}{a} \tag{1}$$

$$\mu_1\left(\frac{\partial v_{\theta 1}}{\partial r}-\frac{v_{\theta 1}}{r}\right)-\mu_2\left(\frac{\partial v_{\theta 2}}{\partial r}-\frac{v_{\theta 2}}{r}\right)+\frac{1}{a}\frac{d\sigma}{dT}\frac{dT_1}{d\theta}=0$$

$$T_1=T_2,\quad \lambda_1\frac{\partial T_1}{\partial r}-\lambda_2\frac{\partial T_2}{\partial r}=0$$

$$r\to\infty,\quad \mathbf{v}_1\to U_\infty\cos\theta\mathbf{e}_r-U_\infty\sin\theta\mathbf{e}_\theta,\quad T_1\to T_\infty$$

$$r=0,\quad |\mathbf{v}_2|<\infty,\quad T_2<\infty$$

The subscripts $i=1, 2$ refer to the external fluid and the drop respectively, U_∞ is the velocity of the approach stream, $\mathbf{g}$ is the gravitational force vector (U_∞ and $|\mathbf{g}|$ must be related such that the total force acting on the drop is equal to zero), $\mathbf{v}_i, p_i, T_i$ and ρ_i are the velocity, pressure, temperature and density, T_∞ is the temperature remote from the drop, $\mu_i, \nu_i, \lambda_i, \chi_i$ are the coefficients of kinematic and dynamic viscosity, thermal conductivity and thermal diffusivity, q is the constant power of the heat sources (sinks) per unit volume of the drop, c_{pi} is the heat capacity, a is the radius of the drop, σ is the surface tension and $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are the unit vectors of the spherical coordinate system.

After introducing a stream function and changing to dimensionless variables, the equations and boundary conditions (1) can be written in the form

$$E^4\psi_i=0,\quad E^2=\frac{\partial^2}{\partial r^2}+\frac{1-\mu^2}{r^2}\frac{\partial^2}{\partial\mu^2},\quad \mu=\cos\theta \tag{2}$$

$$r=1,\quad \psi_1=\psi_2=0,\quad \partial\psi_1/\partial r=\partial\psi_2/\partial r$$

$$r\to\infty,\quad \psi_1\to r^2(1-\mu^2)/2,\quad r=0,\quad \psi_2/r^2<\infty \tag{3}$$

$$r=1,\quad \left(2\frac{\partial}{\partial r}-\frac{\partial^2}{\partial r^2}\right)(\psi_1-\beta\psi_2)=M(1-\mu^2)\frac{\partial\varphi_1}{\partial\mu} \tag{4}$$

$$\frac{\mathrm{Pe}}{r^2}\frac{\partial(\psi_1,\varphi_1)}{\partial(r,\mu)}=\Delta\varphi_1$$

$$\frac{\varkappa^{-1}\mathrm{Pe}}{r^2}\frac{\partial(\psi_2,\varphi_2)}{\partial(r,\mu)}=\Delta\varphi_2+\delta^{-1} \tag{5}$$

$$r=1,\quad \varphi_1=\varphi_2\quad \partial\varphi_1/\partial r=\delta\partial\varphi_2/\partial r$$

$$r\to\infty,\quad \varphi_1\to 0,\quad r=0,\quad \varphi_2<\infty \tag{6}$$

$$v_{ri}=\frac{1}{r^2\sin\theta}\frac{\partial\psi_i}{\partial\theta},\quad v_{\theta i}=-\frac{1}{r\sin\theta}\frac{\partial\psi_i}{\partial\theta}$$

$$\mathrm{Pe}=\frac{U_\infty a}{\chi_1},\quad \varkappa=\frac{\chi_2}{\chi_1},\quad \beta=\frac{\mu_2}{\mu_1},\quad \delta=\frac{\lambda_2}{\lambda_1}$$

$$\varphi_i=\frac{\lambda_1(T_i-T_\infty)}{qa^2},\quad M=\frac{qa^2}{\lambda_1\mu_1U_\infty}\frac{d\sigma}{dT}$$

The distance is referred to a, the stream function to $U_\infty a^2$ and M is the Marangoni number.

The solution of problem (2) with the boundary conditions (3) has the form /8/

$$\psi_1=\left(r^2+Ar-\frac{A+1}{r}\right)\frac{1-\mu^2}{2}+\sum_{n=3}^{\infty}A_n(r^{-n+3}-r^{-n+1})G_n(\mu)$$

$$\psi_2=\left(A+\frac{3}{2}\right)(r^4-r^2)\frac{1-\mu^2}{2}+\sum_{n=3}^{\infty}A_n(r^{n+2}-r^n)G_n(\mu) \tag{7}$$

Here, $G_n(\mu)$ is an n-th order Gegenbauer function of the first kind of degree $-1/_2$. The constants $A, A_3, A_4, \ldots$ still remain undetermined and must be found from condition (4) when solving the problem of the temperature distribution.

By assuming that the numbers Pe and $\varkappa^{-1}\mathrm{Pe}$ are small, we find an approximate solution of the thermal problem by the method of matching asymptotic expansions (/9/, for example), which were confined to the zeroth- and first-order terms. The solution in the external domain, the internal domain and within the drop is represented respectively in the following form:

$$\varphi_1=\varphi_1^{(0)}+\mathrm{Pe}\varphi^{(1)}{}_1,\quad \varphi_1=\varphi_{10}+\mathrm{Pe}\varphi_{11},\quad \varphi_2=\varphi_{20}+\mathrm{Pe}\varphi_{21} \tag{8}$$

By successively determining the zeroth and first terms of expansions (8) from Eqs.(5) in which the stream functions are specified by expressions (7) with the boundary conditions

(6) and using the condition for the matching of the internal and external expansions, we obtain

$$\varphi_{10} = \frac{1}{3r}, \quad \varphi_{20} = \frac{1}{3}\left(1 + \frac{1}{2\delta}(1 - r^2)\right)$$
$$\varphi_1^{(0)} = 0, \quad \mathrm{Pe}\,\varphi_1^{(1)} = \frac{1}{3r}\exp\frac{\mathrm{Pe}\, r(\mu - 1)}{2} \tag{9}$$

$$\varphi_{11} = -\frac{1}{6} + \frac{1}{3}\left\{\frac{1}{2}\left(1 + \frac{A}{r} + \frac{A+1}{2r^3}\right) - (\delta + 2)^{-1}\left[\frac{1}{4}(5A + 3) + \frac{3}{4}\delta(A + 1) + \frac{2}{35}\varkappa^{-1}\left(\frac{3}{2} + A\right)\right]\frac{1}{r^3}\right\}\mu + \sum_{n=2}^{\infty}\left\{K_{n1}\frac{1}{r^{n+1}} + \frac{1}{6}\left(\frac{1}{(n+1)r^{n+2}} + \frac{1}{nr^n}\right)\right\}A_{n+1}P_n(\mu)$$
$$\varphi_{21} = -\frac{1}{6} + \frac{1}{3}\left\{\left[\frac{1}{4}(A + 3) - \varkappa^{-1}\left(\frac{3}{2} + A\right)\frac{18\delta^{-1} + 17}{140}\right](2 + \delta)^{-1} r + \varkappa^{-1}\left(\frac{3}{2} + A\right)\delta^{-1}\left(\frac{1}{10}r^3 - \frac{1}{28}r^5\right)\right\}\mu + \sum_{n=2}^{\infty}\left\{K_{n2}r^n - \frac{\delta^{-1}\varkappa^{-1}}{6}\left(\frac{r^{n+4}}{4n+10} - \frac{r^{n+2}}{2n+3}\right)\right\}A_{n+1}P_n(\mu) \tag{10}$$

Here, $P_n(\mu)$ is a Legendre polynomial of the first kind of degree n and K_{n1} and K_{n2} are expressions, dependent on n, δ and $\varkappa$ which are not given here on account of their unwieldiness.

By substituting expressions (7), (9) and (10) in (4), we obtain the values of the constants A, A_n:

$$A = -\left[m\left(\frac{3}{4} - \frac{3}{35}\varkappa^{-1}\right) + 1 + \frac{3}{2}\beta\right]\left[m\left(\frac{1}{4} - \frac{2}{35}\varkappa^{-1}\right) + 1 + \beta\right]^{-1}$$
$$m = -\frac{M\,\mathrm{Pe}}{9(\delta + 2)}, \quad A_n = 0, \quad n = 3, 4, \ldots \tag{11}$$

The nature of the dependence of A on the parameter m is shown in Fig.1. Depending on the value of the parameter $\varkappa^{-1}$, the asymptotes of the graph may move into other half planes.

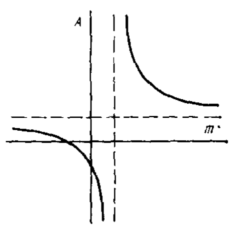

Fig.1

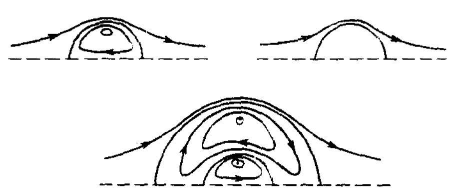

Fig.2

Since the surface tension usually falls as the temperature increases, $m > 0$ corresponds to the evolution of heat while $m < 0$ corresponds to the absorption of heat.

The force acting on the drop due to the viscous and thermocapillary stresses is equal to /8/

$$F = -4\pi\mu_1 a A U_\infty \tag{12}$$

We note that the force (12) has been calculated under the assumption of uniform motion of the drop which is only possible in the case when the total force acting on the drop is equal to zero. Since the force (12) is proportional to the velocity and vanishes together with it, it is necessary, in order to realize the case of uniform motion of the drop, to assume that there is a certain lateral force which counterbalances the force (12).

Depending on the values of the parameters, the force (12) may be both a resistive force (in this case a force is necessary in order to maintain the motion, such as a gravitational force acting in the direction of the velocity), as well as a tractive force (whereupon a force acting in a direction opposite to the velocity must exist in order to maintain the motion).

Another interpretation of the results obtained is also possible when an external force, such as a gravitational force, acting on the drop is specified. In this case, the velocity of motion of the drop can be found from the equality of the force (12) to the external field.

Different qualitative pictures of the circumfluence of the drop (Fig.2) correspond to different values of the constant A. When $A > -^3/_2$, the picture of the flow around the drop is analogous to a Rybchinskii-Hadamard circumfluence. As the magnitude of A becomes smaller, the intensity of the circulation of the fluid within the drop falls off and vanishes when $A = -^3/_2$. As A becomes still smaller $(A < -^3/_2)$ a circulation zone forms around the drop and the external radius of this zone is given by the relationship

$$r_c = {}^1/_2 \mid A + 1 \mid [1 + (1 + 4/\mid A + 1 \mid)^{1/2}]$$

The direction of the internal circulation becomes opposite to the corresponding direction in the case of a Rybchinskii-Hadamard case. At the same time, as follows from (12), the resistive force acting on the drop exceeds the Stokes force acting on a hard sphere.

In the limiting case when $\beta \to \infty$ (the substance of which the drop is comprised has a high viscosity), the thermocapillary effect has no influence on the motion, $A \to -^3/_2$, and there will be a flow around the drop as if it were a solid sphere and Stokes law is obtained from (12). When $M = 0$ (the absence of heat sources or when the surface tension is independent of temperature), or $\delta \to \infty$ (a high thermal conductivity of the substance constituting the drop), there is no thermocapillary effect since the temperature is constant along the surface and the usual expressions for the circumfluence of a drop and the resistive forces acting on it /8/ are obtained from (7) and (12), taking account of (11).

We note that relationships (11) for constant A_n $(n = 3, 4, \ldots)$ are not universal in the sense that, for each number $n = 3, 4, \ldots$, there are isolated values of the parameter m (we shall denote them by m_n, $n = 3, 4, \ldots$) for which the corresponding constant A_n may not be equal to zero. When $m = m_n$, the algebraic equation for determining the constant A_n corresponding to this number is reduced to the form $0 \cdot A_n = 0$ and the constant A_n can take arbitrary values. Further investigation of this question is only possible on the basis of higher order approximations with respect to the Reynolds and Péclet numbers.

At certain values of the parameter $m = m_*$ and $m = m_{**}$, the constant A may respectively vanish or become infinite. If $A = 0$, the force acting on the drop due to viscous and thermocapillary stresses vanishes and, in the approximation considered here, the analysis which has been presented is only correct in the case when there is no gravitation, when the velocity of the motion may be arbitrary, and in order to obtain its actual value it is necessary to carry out the analysis using higher-order approximations. If a non-zero mass force is specified, the analysis which has been carried out will be incorrect in the case of values of the parameter m from a certain neighbourhood of the value $m = m_*$. As the magnitude of the mass force decreases, this neighbourhood also becomes smaller and, in the limit, contracts to a point.

If $A \to \infty$ $(m = m_{**})$, then, in any finite gravitational field, the drop will remain at rest (that is, $U_\infty = 0$) and, at the same time, the flow outside of and within the drop is represented in dimensional variables in the form

$$\psi_1 = \gamma \left(\frac{r}{a} - \frac{a}{r}\right)(1 - \mu^2), \quad \psi_2 = \gamma \left(\left(\frac{r}{a}\right)^4 - \left(\frac{r}{a}\right)^2\right)(1 - \mu^2)$$

$$\gamma = g\,(\rho_2 - \rho_1)\,a^4/6\mu_1$$

Here g is the magnitude of the gravitational acceleration ($g > 0$, if the gravitational force is directed along the axis from which the angle θ is measured, and $g < 0$ otherwise).

As a result of this motion a force acts on the drop which compensates the mass force. In the other approach to the formulation of the problem, when the velocity U_∞ is assumed to be specified and the mass force required to maintain the motion is calculated from relationship (12), the analysis carried out in this paper will be incorrect for values of the parameter m from a certain neighbourhood of the value of $m = m_{**}$. As the velocity decreases, this neighbourhood also becomes smaller and, in the limit, contracts to a point. The remarks concerned with the vanishing of the constants A, A_n $(n = 3, 4, \ldots)$ and the constant A becoming infinite may also pertain to /5, 6, 10/.

In the formulation of problem (2)-(6), the boundary condition for the normal stresses, which was not required in order to find the flow around the drop, was omitted. The pressure distribution is also obtained but no use is made of it /8/. It can be shown that, on substituting already known solutions for the stream function and pressure, this boundary condition becomes an indentity. This means that, in the approximation which is being considered, the drop remains strictly spherical and an analysis of its shape would have to be carried out using higher-order approximations.

The presence of a tractive force which is proportional to the velocity may serve as an indication of the instability of the state of rest or the uniform motion of the drop or, in other words, at those values of the parameters of the problem at which the constant A is positive, the stationary regimes of the motion which have been found will apparently be unstable. At the same time it is possible to postulate the existence of other stationary regimes of the motion which, fundamentally, are not found within the framework of the Stokes nature of the flow and the smallness of the numbers Pe and $\varkappa^{-1}$ Pe (except when the constant A is close to zero) and, in order to find them, it is necessary to get rid of at least one of these assumptions. In particular, it follows from this that a drop which is initially at rest in the absence of gravitation cannot maintain this state of rest, and begins to drift in a random direction.

In essence the stability of the mode which has been considered above and corresponds to a Gegenbauer function $G_2(\mu)$ (in the expressions for the stream function (7)) and to a Legendre polynomial $P_1(\mu)$ (in expressions for the temperature (10)), since it only makes a contribution to the force (12) and affects the translational motion of the drop. However, it should be kept in mind that the instability can also develop in higher-order modes ($G_{n+1}(\mu)$ and $P_n(\mu)$, $n \geqslant 2$). Evidence for this comes, in particular, from the fact that the coefficients A_n $(n \geqslant 3)$ differ from zero for certain values of the parameters of the problem and, generally speaking, the negativeness of the constant A is not sufficient but only necessary for stability. A more detailed discussion of these questions lies outside of the scope of this paper.

In concluding we shall indicate two further problems on the thermocapillary motion of a drop, the solutions of which, using the simplifying assumptions adopted here, can be directly established from the results which have been obtained above. Let us consider the thermocapillary motion of a drop when the heat sources (sinks) of constant intensity J are not located in the bulk but on the surface of the drop. It is readily shown that the results corresponding to this case are obtained if one formally puts

$$\varkappa^{-1} = 0, \quad q = 3J/a$$

in all of the relationships referring to the case of homogeneous internal heat evolution.

Let us also consider the thermocapillary motion of a drop when the heat sources (sinks) of constant power are located not within but outside of the drop. Although the temperature field will be non-stationary with such a distribution of heat sources, it is readily established by making the substitution $T_i' = T_i - T_0 - (q'/(c_{p_2}\rho_{p_2}))(t - t_0)$ in the initial system of equations which takes account of the homogeneous heating of the whole medium that the results which have previously been obtained are also applicable in this case with the sole difference that it is now necessary to understand the quantity q as

$$q = -\alpha q', \quad \alpha = c_{p_2}\rho_{p_2}/(c_1\rho_1)$$

Here T_0 is the temperature remote from the drop at a certain instant of time t_0, t is the time and q' is the constant power of the heat sources (sinks) per unit volume of the external field.

REFERENCES

1. YOUNG N.O., GOLDSTEIN J.S. and BLOCK M.G., The motion of bubbles in a vertical temperature gradient, J. Fluid Mech., 6, 3, 1959.
2. BRATUKHIN YU.K., Thermocapillary drift of a droplet of a viscous fluid, Izv. Akad. Nauk SSSR, Mekh. Zhid. i Gaz., 5, 1975.
3. YALAMOV YU.I. and SANASARYAN A.S., The motion of a drop in a viscous medium with temperature inhomogeneity, Inzh.-Fiz., Zh., 28, 6, 1975.
4. ANTANOVSKII L.K. and KOPBOSYNOV B.K., Non-stationary thermocapillary drift of a drop of a viscous fluid, Prikl. Mekh. i Tekhn., Fiz., 2, 1986.
5. RYAZANTSEV YU.S., On the thermocapillary motion of a reacting drop in a chemically active medium, Izv. Akad. Nauk SSSR, Mekh., Zhid. i Gas., 3, 1985.
6. GOLOVIN A.A., On the effect of surface tension on the motion of reacting drops, in: Elementary Processes in Chemically Reaction Media, Fiz. Tekhn. Inst. (MFTI), Moscow, 1985.
7. GOLOVIN A.A., GUPALO YU.P. and RYAZANTSEV YU.S., On the chemothermocapillary effect during the motion of a drop in a fluid, Dokl. Akad. Nauk SSSR, 290, 1, 1986.
8. HAPPEL J. and BRENNER H., Low Reynolds Number Hydrodynamics, Mir, Moscow, 1976.
9. GUPALO YU.P., POLYANIN A.D. and RYAZANTSEV YU.S., Mass and Heat Exchange between Reacting Particles and a Flow, Nauka, Moscow, 1985.
10. GOLOVIN A.A., GUPALO YU.P. and RYAZANTSEV YU.S., Chemoconcentration capillary effect during the motion of a drop in a fluid, Izv. Akad. Nauk SSSR, Mekh. Zhid. i Gaz., 1, 1988.

Translated by E.L.S.